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The Decrease in the Overall Algorithmic Complexity of the Spin-Echo Effect

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Recently Lloyd and Zurek studied the algorithmic complexity of the spin-echo effect and concluded that the overall complexity of spins together with the magnetic field grew slowly even during the rephasing stage. In this paper we show that, in contrast to their conclusion, the complexity decreases during the rephasing stage. We also clarify the origin of the disagreement.

KEY WORDS: Algorithmic entropy; complexity; randomness; spin-echo effect.

1. INTRODUCTION

Recently Lloyd and Zurek[1] investigated the spin-echo effect[2-5] by means of statistical entropy and algorithmic complexity.[6,7] They studied the following idealized model[1,4,5]: The system consists of a magnetic field $\mathbf{H}_0$ and N ($\gg 1$) spins. The magnetic field is static and points along the z axis. In the ground state the spins also point along the z axis. At a certain time (say, at $t=0$) the spins are aligned along the x axis by a resonant radio-frequency pulse ("$\pi/2$ pulse"), and then the ith spin begins to precess in the x–y plane at the Larmor frequency ω_i which is determined by the strength of the magnetic field at its site. Hereafter we refer to the time $t=0$ as the initial time and the state at $t=0$ as the initial state. Since the Larmor frequencies $\{\omega_i\}$ of spins vary slightly around a value ω_0 because of the inhomogeneity of the magnetic field, the precession angles $\{\phi_i(t)\}$ become different as t increases. At a time t_r a second pulse ("π pulse") conjugates their phases. Because individual spins keep precessing at their initial frequencies ω_i, all spins converge back along *one* direction, say ϕ_{echo}, at

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time $2t_r$. (The spin–spin and spin–lattice relaxation times are assumed to be longer than $2t_r$.)

One of their most remarkable conclusions is that although the algorithmic complexity of the state of the spins rapidly increases during $0 < t < t_r$, and then decreases during $t_r < t < 2t_r$, the overall complexity $K(\{\phi_i(t)\}, \{\omega_i\})$ of spins together with the magnetic field grows logarithmically as a function of the elapsed time:

$$\begin{aligned} K(\{\phi_i(t)\}, \{\omega_i\}) &\equiv K(\{\lfloor \phi_i(t)/\Delta\phi \rfloor\}, \{\lfloor \omega_i/\Delta\omega \rfloor\}) \\ &= K(\{\lfloor \omega_i/\Delta\omega \rfloor\}) + K(\lfloor \omega_0 t/\Delta\phi \rfloor) + \text{const} \end{aligned} \tag{1.1}$$

where $\lfloor x \rfloor$ is the greatest integer which is less than or equal to x, $\Delta\omega$ (resp. $\Delta\phi$) is the accuracy within which the Larmor frequencies (resp. the angles of spins) are determined, and $K(n)$ [resp. $K(\{n_i\})$] is the complexity of an integer n (resp. a list of integers $\{n_i\}$). Since $K(n)$ is approximated by $\log_2 |n|$, Eq. (1.1) indicates that $K(\{\phi_i(t)\}, \{\omega_i\})$ increases as the logarithm of t.

This conclusion, however, seems contrary to one's intuition. Because the complexity of the state of the system is defined as the length of the shortest program that can print out the description of the state,[6,7] it should be determined once the state is given (provided that the accuracy of the description is fixed). Moreover, at time $2t_r$ the system returns to the initial state in the sense that the spins are all lined up again and the magnetic field does not change from the initial state at all. Therefore the complexity should also return close to the initial value (i.e., the value at $t = 0$) at time $2t_r$. In fact, since the description of the state of this system at time t is given by the list $\{\phi_1(t), \phi_2(t), ..., \phi_N(t), \omega_1, \omega_2, ..., \omega_N\}$, it can be printed out by a program that includes only ϕ_{echo} and $\{\omega_i\}$ as a data part at time $2t_r$ because $\phi_i(2t_r) = \phi_{\text{echo}}$ for all i. Therefore an upper bound of the overall complexity $K(\{\phi_i(2t_r)\}, \{\omega_i\})$ is given by

$$K(\{\lfloor \omega_i/\Delta\omega \rfloor\}) + K(\lfloor \phi_{\text{echo}}/\Delta\phi \rfloor) + C_1 \tag{1.2}$$

where C_1 is the length of the program outside the data part and depends on neither t nor N. The value of Eq. (1.2) is smaller than that of Eq. (1.1) provided that $\omega_0 t > \phi_{\text{echo}}$, that is, except the very early time.

In the next section we investigate the algorithmic complexity of this ideal spin-echo phenomenon near time $2t_r$ and show that the overall complexity of spins together with the magnetic field decreases as the logarithm of $2t_r - t$. The origin of the disagreement with Lloyd and Zurek[1] is discussed in Section 3.

2. COMPLEXITY NEAR TIME $2t_r$

The way to estimate the complexity is simple: During $t > t_r$, the angle $\phi_i(t)$ of the ith spin is given by

$$\phi_i(t) = \phi_{\text{echo}} + \omega_i(t - 2t_r) \tag{2.1}$$

It is obvious that one can calculate $\{\phi_i(t)\}$ from the data ϕ_{echo}, $\{\omega_i\}$, and $t - 2t_r$ by using this equation. (Note that it is $t - 2t_r$, not t, that is regarded as data. We discuss this point in the next section.) Therefore an upper bound of the complexity is given by the length of the program that calculates $\{\phi_i(t)\}$ by using these data, which is equal to the sum of the complexities of these data and the length C_2 of the part of the program that specifies the algorithm for the calculation. The complexities of ϕ_{echo}, $\{\omega_i\}$, and $t - 2t_r$ are given by $K(\lfloor \phi_{\text{echo}}/\varepsilon_\phi \rfloor)$, $K(\{\lfloor \omega_i/\Delta\omega \rfloor\})$, and $K(\lfloor (t - 2t_r)/\varepsilon_t \rfloor)$, respectively, where ε_ϕ (resp. ε_t) is the accuracy of ϕ_{echo} (resp. $t - 2t_r$). Note that they should satisfy

$$\varepsilon_\phi + \omega_i \varepsilon_t + \Delta\omega \, |t - 2t_r| < \Delta\phi \qquad \text{for all } i \tag{2.2}$$

to make the computational error of Eq. (2.1) less than $\Delta\phi$. Thus,

$$K(\{\phi_i(t)\}, \{\omega_i\}) \leqslant K(\lfloor \phi_{\text{echo}}/\varepsilon_\phi \rfloor) + K(\{\lfloor \omega_i/\Delta\omega \rfloor\}) + K(\lfloor (t - 2t_r)/\varepsilon_t \rfloor) + C_2 \tag{2.3}$$

for t satisfying Eq. (2.2). For most times t in this period it is reasonable to estimate $K(\{\phi_i(t)\}, \{\omega_i\})$ by using the right-hand side of Eq. (2.3) because $K(\{\lfloor \omega_i/\Delta\omega \rfloor\}) \leqslant K(\{\phi_i(t)\}, \{\omega_i\}) + O(1)$,[6] and Eq. (2.1) is so simple that the algorithm for the calculation is very short. Moreover, since $K(\lfloor (t - 2t_r)/\varepsilon_t \rfloor)$ is approximated by $\log_2 \lfloor |t - 2t_r|/\varepsilon_t \rfloor$, $K(\{\phi_i(t)\}, \{\omega_i\})$ *decreases* as the logarithm of $2t_r - t$.

3. SUMMARY AND DISCUSSIONS

We have studied the algorithmic complexity of the spin-echo effect and concluded that the overall complexity decreases near time $2t_r$ in contrast to the result of Lloyd and Zurek.[1]

Where does this disagreement originate? First, the essential difference between their estimate Eq. (1.1) and our Eq. (2.3) is that $t - 2t_r$ is replaced with t in their equation, i.e., in their estimation t (with the appropriate accuracy) is required as a part of data from which the description of the state at time t is calculated. It is clear from Eq. (2.1), however, that it is $t - 2t_r$, not t, that is necessary. For example, when $t = 10{,}000\varepsilon_t$ and

$2t_r = 10{,}001\varepsilon_t$, two numbers 10,000 and 10,001 are not necessary as long as -1 $(=10{,}000 - 10{,}001)$ is given. The reader may consider that there is no way to obtain $t - 2t_r$ other than by calculating it from t and t_r, and eventually both of them must be included properly in the program. However, there is no reason to forbid calculating $t - 2t_r$ beforehand and then incorporating the answer in the program. In short, the value of $t - 2t_r$ surely *exists* on its own, independent of the *way* to obtain it, and *therefore the program including it also exists.* Consequently, it is valid to take the length of it as an upper bound of the complexity.

Here we note that in our discussion the complexity of the *generator of the* $\pi/2$ *pulse* is not considered, because we are dealing with the complexity of spins and the magnetic field as in the discussion of Lloyd and Zurek. If one wishes to discuss the complexity of the whole system (spins + magnetic field + generator), it should be noted that the generator must include t_r as data which specify the time of reversal, and consequently the complexity includes a term of the form $\log t_r$. This is reflected in the fact discussed in ref. 1 that an observer who wishes to take advantage of the energy in the echo pulse must possess the algorithmic information necessary to specify t_r.

Next, let us discuss subtle problems of the estimation of Lloyd and Zurek. Their estimation is based on the idea that because $\{\phi_i(t)\}$ can be calculated from $\{\omega_i\}$ and the value of t, the length of the program which includes them and does the calculation can be used as an upper bound of the complexity. Although there is a better estimation near time $2t_r$ as shown above, their idea itself is natural and expected to give good estimates of the complexity at small t. However, they used their idea somewhat confusingly. Therefore it would be worthwhile to discuss the problems and restudy the complexity at small t. The point at issue is $\Delta\omega$ in Eq. (1.1). Lloyd and Zurek supposed that the program should be able to calculate $\{\phi_i(t)\}$ within the accuracy $\Delta\phi$ up to a time $t_{\max}$ which is greater than any relevant time, and consequently they deduced that $\Delta\omega$ should be $\Delta\phi/t_{\max}$. Thus, they gave the following estimate:

$$K(\{\lfloor \omega_i t_{\max}/\Delta\phi \rfloor\}) + K(\lfloor \omega_0 t/\Delta\phi \rfloor) + \text{const} \tag{3.1}$$

including a large constant proportional to $N \log t_{\max}$ in the first term. However, there are two problems in their discussion. The first is the meaning of $\Delta\omega$. Estimating the *overall* complexity, one should consider that $\{\omega_i\}$ play dual roles, one as part of the description of the state and the other as part of the data from which $\{\phi_i(t)\}$ are calculated. Although the accuracy necessary for the former role (referred to as $\Delta_d\omega$ hereafter) and that necessary for the latter role $(\Delta_c\omega)$ are quit distinct, they seem to be mixed up with each other in their discussion. While $\Delta_c\omega$ is determined in order that $\{\phi_i(t)\}$ can be calculated within the accuracy $\Delta\phi$, $\Delta_d\omega$ should be determined in a similar way as that in which $\Delta\phi$ is determined, that is, by the power of measuring devices. Consequently, the program should include $\{\omega_i\}$, whose accuracy $\Delta_p\omega$ is equal to the lesser of $\Delta_d\omega$ and $\Delta_c\omega$. The second problem is about $t_{\max}$. To obtain the description at $t < t_{\max}$, it is not necessary to demand that the program should calculate $\{\phi_i(t)\}$ up to time $t_{\max}$; it is enough to calculate them up to only t itself. Therefore their assumption that the program should calculate the state within the accuracy up to $t_{\max}$ is too strict and makes the program too long.

The estimation for the complexity at time t proceeds as follows: Because $\Delta_c\omega$ is given by $\Delta\phi/t$, $\Delta_p\omega$ is given by

$$\Delta_p\omega = \min\{\Delta_c\omega, \Delta_d\omega\} = \begin{cases} \Delta_d\omega & \text{when} \quad t < \Delta\phi/\Delta_d\omega \\ \Delta\phi/t & \text{when} \quad t > \Delta\phi/\Delta_d\omega \end{cases} \tag{3.2}$$

By substituting above $\Delta_p\omega$ as $\Delta\omega$ into Eq. (1.1), we obtain the upper bound of the complexity given by

$$\begin{aligned} &K(\{\lfloor \omega_i/\Delta_d\omega \rfloor\}) + K(\lfloor \omega_0 t/\Delta\phi \rfloor) + \text{const} && \text{when} \quad t < \Delta\phi/\Delta_d\omega \\ &K(\{\lfloor \omega_i t/\Delta\phi \rfloor\}) + K(\lfloor \omega_0 t/\Delta\phi \rfloor) + \text{const} && \text{when} \quad t > \Delta\phi/\Delta_d\omega \end{aligned} \tag{3.3}$$

Note that the second expression of (3.3) grows by $N \log t$ rather than by $\log t$. This is reasonable, because as t grows, the accuracy of the initial condition included in the algorithm must grow to give $\{\phi_i(t)\}$ within the given precision $\Delta_d\omega$, so that the length of the algorithm also grows.

Let us return to the complexity during the rephasing stage. Using the above notation, $\Delta\omega$ in Section 2 should also be written as $\Delta_d\omega$. The essential point is that although their idea is natural, there is a better estimation of the complexity near time $2t_r$. Once again we note that our estimation in Section 2 is based on the same equation (2.1) as the one that Lloyd and Zurek used. The only difference is, as we have discussed, that in Section 2 the expression $t - 2t_r$ included in Eq. (2.1) is regarded as one number which exists on its own. This suggests that to estimate the complexity of a physical state, besides searching for an equation by which the state can be calculated, one has to consider how to "interpret" the equation to make the program solving the equation as short as possible.

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